

Convergence of an approximation for rotationally symmetric two-phase lipid bilayer membranes

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Abstract. We consider a diffuse interface approximation for the lipid phases of rotationally symmetric two-phase bilayer membranes and rigorously derive its Γ -limit. In particular, we prove that limit vesicles are C^1 across interfaces, which justifies a regularity assumption that is widely made in formal asymptotic and numerical studies. Moreover, a limit membrane may consist of several topological spheres, which are connected at the axis of revolution and resemble complete buds of the vesicle.

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1 Introduction

Lipid bilayer membranes are an integral part of many biological systems and display a rich variety of shapes and shape transformations; in particular, membranes that consist of two or more lipid phases have a complex morphology affected by the interplay of elastic properties and phase separation [21, 9, 2]. The spontaneous curvature model for two-phase lipid bilayer vesicles describes equilibrium shapes as surfaces minimising the energy

$$\sum_{j=\pm} \int_{M^j} k^j (H - H_s^j)^2 + k_G^j K d\mu + \sigma \mathcal{H}^1(\partial M^+) \quad (1.1)$$

among all closed surfaces $M = M^+ \cup M^-$, $M^+ \cap M^- = \emptyset$ with prescribed areas for M^\pm [6, 12, 13, 15, 22]. Here H and K are the mean curvature and the Gauss curvature of the membrane surface M , and μ is its area measure. The bending rigidities $k^\pm > 0$ and the Gauss rigidities k_G^\pm are elastic material parameters, and H_s^\pm – the so-called spontaneous curvatures – are supposed to reflect an asymmetry in the membrane. In the simplest case, the rigidities and spontaneous curvatures are constant within each lipid phase but different between the two phases. The length of the phase interfaces $\partial M^+ = \partial M^-$ is denoted by $\mathcal{H}^1(\partial M^+)$, and σ is a line tension parameter.

In [15] the Euler-Lagrange equations of (1.1) for axially symmetric two-phase membranes with exactly one interface are studied. The authors mention the possibility of different smoothness conditions at the interface, their analysis, however, is done for smooth membranes, which are C^1 across the interface, only. Phase field models for the lipid phases and also the membrane surface are introduced in [10, 11, 28, 17] and studied numerically; convergence to the sharp interface limit is obtained by asymptotic expansion and under additional smoothness assumptions and topological restrictions.

In this paper we are interested in the convergence of a diffuse interface approximation for the lipid phases in a rotationally symmetric setting without imposing smoothness or the topological structure of limit vesicles in advance. More precisely, for a closed surface

M_γ obtained by rotating a curve γ and an associated rotationally symmetric phase field $u : M_\gamma \rightarrow \mathbb{R}$ we consider the approximate energy

$$\int_{M_\gamma} k(u) (H - H_s(u))^2 + k_G(u) K \, d\mu + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) \, d\mu. \quad (1.2)$$

Here the second integral, where W is a standard double well potential such as $W(u) = (1 - u^2)^2$, is the diffuse interface energy from the Cahn-Hilliard theory of phase transitions [5] in the setting of surfaces. As $\varepsilon \rightarrow 0$, the phase field is forced to ± 1 , hence the first integral in (1.2) resembles the curvature integral in (1.1), provided that $k(u)$, $k_G(u)$ and $H_s(u)$ are extensions of the given parameters k^\pm , k_G^\pm and H_s^\pm . We prove that, under certain restrictions on these parameters, the Γ -limit of (1.2) is given by (1.1) for rotationally symmetric membranes. In particular, we obtain that sequences $(\gamma_\varepsilon, u_\varepsilon)$ with uniformly bounded energy have a subsequence that converges to a limit membrane consisting of finitely many regular topological spheres, which are connected at the axis of revolution. By our assumption on the parameters and the approximation procedure, the limit model has the property that membranes are C^1 across interfaces. For an approach that allows tangent singularities at interfaces in the limit see [14].

Equi-coercivity and Γ -convergence also yield the existence of a minimiser for the limit model. Upon completion of this work, we became aware of the preprint [7], where the existence of energy-minimal two-phase membranes in a setting similar to our limit model is studied and similar issues as in our equi-coercivity and lower bound arguments are addressed.

The paper is organised as follows. Section 2 recalls some facts about surfaces of revolution, in Section 3 we present our setting and state the convergence theorem. We prove the theorem in Section 4 and conclude with some remarks on generalisations in Section 5.

2 Surfaces of revolution

2.1 Basic definitions and notation

Let $I \subset \mathbb{R}$ be an open bounded interval and $\gamma = (x, y) : I \rightarrow \mathbb{R}^2$ a Lipschitz parametrised curve in the upper half of the xy -plane, that is, $y(t) \geq 0$ for all $t \in I$. We denote by M_γ the surface in \mathbb{R}^3 obtained by rotating γ about the x -axis, thus M_γ is the image of $\bar{I} \times [0, 2\pi)$ under the Lipschitz continuous map

$$\Phi : (t, \theta) \mapsto (x(t), y(t) \cos \theta, y(t) \sin \theta);$$

γ is called generating curve of M_γ . See [8, 16] for a detailed discussion of surfaces.

Since γ is Lipschitz continuous, it is weakly and almost everywhere differentiable with bounded derivative γ' , and the Fundamental Theorem of Calculus

$$\gamma(t_1) - \gamma(t_0) = \int_{t_0}^{t_1} \gamma'(t) \, dt$$

holds for all $t_0, t_1 \in \bar{I}$. The length of γ is given by

$$\mathcal{L}_\gamma = \int_I |\gamma'(t)| \, dt,$$

and after removing at most countably many constancy intervals, pulling holes together and reparametrising, we may assume that γ is parametrised with constant speed $|\gamma'| \equiv \mathcal{L}_\gamma / |I| =: q_\gamma$ almost everywhere in I [4, Lemma 5.23].

By μ we denote the area measure of M_γ , that is $d\mu = |\partial_t \Phi \wedge \partial_\theta \Phi| dt d\theta = |\gamma'| y dt d\theta$, and we write

$$\mathcal{A}_\gamma = \int_{M_\gamma} d\mu = 2\pi \int_I |\gamma'| y dt$$

for the area of M_γ . Moreover, for a measurable subset J of I , we let $M_\gamma(J)$ be the part of M_γ that is obtained by rotating the curve segment $\gamma(J)$, and refer to the corresponding length and area as $\mathcal{L}_\gamma(J)$ and $\mathcal{A}_\gamma(J)$, respectively. If γ is embedded, then also M_γ is, and μ is the two-dimensional Hausdorff measure \mathcal{H}^2 restricted to M_γ ; in general, however, the multiplicity of μ may be larger than 1.

The tangent space $\mathcal{T}_{(t_0, \theta_0)} M_\gamma$ exists for almost every $(t, \theta) \in I \times [0, 2\pi)$ and is the plane spanned by the orthonormal vectors

$$\xi_1 = \frac{\partial_t \Phi}{|\partial_t \Phi|} = \frac{1}{|\gamma'|} (x', y' \cos \theta, y' \sin \theta) \quad \text{and} \quad \xi_2 = \frac{\partial_\theta \Phi}{|\partial_\theta \Phi|} = (0, -\sin \theta, \cos \theta); \quad (2.1)$$

a unit normal is given by

$$\nu = \frac{\partial_t \Phi \wedge \partial_\theta \Phi}{|\partial_t \Phi \wedge \partial_\theta \Phi|} = \frac{1}{|\gamma'|} (-y', x' \cos \theta, x' \sin \theta). \quad (2.2)$$

We associate tangent space, normal and all other geometric quantities to the parameter (t, θ) and not to the point $\Phi(t, \theta)$ on the surface M_γ , because M_γ is not necessarily embedded. For the same reason, we consider a function $f: M_\gamma \rightarrow \mathbb{R}^k$ to be a function $F: \bar{I} \times [0, 2\pi) \rightarrow \mathbb{R}^k$ of the parameters; on embedded parts of M_γ this amounts to $f(\Phi(t, \theta)) = F(t, \theta)$. Given a tangent vector ξ at $(t_0, \theta_0) \in I \times (0, 2\pi)$, the directional derivative of f in direction ξ is defined as

$$D_\xi f(t_0, \theta_0) = \left. \frac{d}{ds} F(\eta(s)) \right|_{s=0},$$

where $\eta: (-\delta, \delta) \rightarrow I \times [0, 2\pi)$ is a C^1 -curve satisfying $\eta(0) = (t_0, \theta_0)$ and $\left. \frac{d}{ds} \Phi(\eta(s)) \right|_{s=0} = \xi$. The tangential gradient of $f: M_\gamma \rightarrow \mathbb{R}$ is

$$\nabla_{M_\gamma} f = (D_{\xi_1} f) \xi_1 + (D_{\xi_2} f) \xi_2,$$

where $\{\xi_1, \xi_2\}$ is any orthonormal basis of the tangent space; see [24, 16] for a detailed discussion. For $\{\xi_1, \xi_2\}$ as in (2.1), we find $D_{\xi_1} f = |\gamma'|^{-1} \partial_t F$ and $D_{\xi_2} f = y^{-1} \partial_\theta F$, hence

$$\nabla_{M_\gamma} f = \frac{1}{|\gamma'|} (\partial_t F) \xi_1 + \frac{1}{y} (\partial_\theta F) \xi_2 = \frac{1}{|\gamma'|^2} (\partial_t F) \partial_t \Phi + \frac{1}{y^2} (\partial_\theta F) \partial_\theta \Phi.$$

In particular, if f is rotationally symmetric, which means that it is independent of θ , then

$$\nabla_{M_\gamma} f(t, \theta) = \frac{F'(t)}{|\gamma'(t)|} \xi_1(t, \theta) \quad \text{and} \quad |\nabla_{M_\gamma} f(t, \theta)| = \frac{|F'(t)|}{|\gamma'(t)|},$$

where $|\cdot|$ is the Euclidean norm in \mathbb{R}^3 .

For the rest of this subsection let $\gamma \in W_{\text{loc}}^{2,1}(I; \mathbb{R}^2)$ be twice weakly differentiable, thus twice differentiable almost everywhere, and $y > 0$ in I . Since ν in (2.2) is weakly differentiable, the shape operator $S: \mathcal{T}_{(t_0, \theta_0)} M \rightarrow \mathcal{T}_{(t_0, \theta_0)} M$, $\zeta \mapsto D_\zeta \nu$ and the second fundamental form $B: \mathcal{T}_{(t_0, \theta_0)} M \times \mathcal{T}_{(t_0, \theta_0)} M \rightarrow \mathbb{R}$, $(\zeta, \xi) \mapsto \xi \cdot D_\zeta \nu$ are well-defined for almost every (t_0, θ_0) . The matrix representation with respect to the basis $\{\xi_1, \xi_2\}$ in (2.1) of both is

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \quad \text{with} \quad \kappa_1 = \frac{-y''x' + y'x''}{|\gamma'|^3} = -\frac{\gamma'' \cdot \gamma'^\perp}{|\gamma'|^3} \quad \text{and} \quad \kappa_2 = \frac{x'}{y|\gamma'|}.$$

The eigenvalues κ_1, κ_2 of S are the principal curvatures of M_γ , and κ_1 is just the signed curvature of γ with respect to the normal $-\gamma'^\perp/|\gamma'| = (y', -x')/|\gamma'|$. The mean curvature H and the Gauss curvature K of M_γ are

$$H = \text{trace } S = \kappa_1 + \kappa_2 \quad \text{and} \quad K = \det S = \kappa_1 \kappa_2.$$

By $|S|^2 = \kappa_1^2 + \kappa_2^2$ we denote the squared Frobenius norm of S , and since $B(\zeta, \zeta) = \zeta \cdot S\zeta$, we also write $|B|^2 = |S|^2$. Obviously, we have $|B|^2 = H^2 - 2K$.

The signs of the principal curvatures and the mean curvature depend on the sign of the normal ν . Our choice above ensures that a unit ball has outer unit normal ν as in (2.2) and curvatures $\kappa_1 = \kappa_2 = +1$ when its generating curve is parametrised “from left to right” such that $x' \geq 0$, for instance by $\gamma(t) = (-\cos t, \sin t)$, $t \in [0, \pi]$.

Let $\varphi: I \rightarrow \mathbb{R}$ be an angle function for γ , that is, let $\varphi(t)$ be the angle between the positive x -axis and the tangent vector $\gamma'(t)$. Since $W_{\text{loc}}^{2,1}$ embeds into C_{loc}^1 , the angle φ can be chosen continuously in I and is then uniquely determined up to multiples of 2π . In terms of φ , the curve γ is characterised by fixing one point and

$$x' = |\gamma'| \cos \varphi, \quad y' = |\gamma'| \sin \varphi.$$

The principal curvatures take the form

$$\kappa_1 = -\frac{\varphi'}{|\gamma'|}, \quad \kappa_2 = \frac{\cos \varphi}{y},$$

and we have

$$K = -\frac{\varphi' \cos \varphi}{|\gamma'| y} = -\frac{(\sin \varphi)'}{|\gamma'| y} = -\frac{(y'/|\gamma'|)'}{|\gamma'| y}. \quad (2.3)$$

From (2.3) we see that for any $J = (a, b) \subseteq I$ the integral

$$\int_{M_\gamma(J)} K d\mu = -2\pi \int_a^b (\sin \varphi)' dt = 2\pi (\sin \varphi(a) - \sin \varphi(b)). \quad (2.4)$$

depends only on the tangent angle at ∂J . If additionally $\varphi \in C^0(\bar{I})$, then (2.4) is by approximation also true for $J = I$, which is just the Gauss Bonnet Theorem for surfaces of revolution. In particular, if $y(\partial I) = \{0\}$ and M_γ is a C^1 -surface, then γ' is perpendicular to the axis of revolution at ∂I and we conclude $\int_{M_\gamma} K d\mu = 4\pi$.

Another consequence of (2.3) is that for γ parametrised with constant speed $q_\gamma > 0$ the integral

$$\int_{M_\gamma(J)} |K| d\mu = \frac{2\pi}{q_\gamma} \int_J |y''| dt \quad (2.5)$$

is the L^1 -norm of y'' . Moreover, in that case we also have $|\gamma''|^2 = \varphi'^2 q_\gamma^2$ and obtain that

$$\int_{M_\gamma(J)} \kappa_1^2 d\mu = \frac{2\pi}{q_\gamma} \int_J |\varphi'|^2 y dt = \frac{2\pi}{q_\gamma^3} \int_J |\gamma''|^2 y dt \quad (2.6)$$

is a weighted L^2 -norm of φ' and γ'' .

If M_γ is a closed surface, that is $y(\partial I) = \{0\}$, κ_2 seemingly degenerates at the axis of revolution. However, if M_γ is sufficiently smooth, the principal curvatures are still well-defined, for instance by taking another local parametrisation of M_γ ; to compute κ_2 in the rotationally symmetric parametrisation, L'Hôpital's rule may be used and yields $\kappa_2 = \kappa_1$ [16].

2.2 Surfaces with L^2 -bounded second fundamental form

The sharp inequality $y > 0$ in I is not conserved by the convergence of curves that our ε -energy yields. If merely $y \geq 0$ in I , the set $\{y > 0\} = \{t \in I : y(t) > 0\}$ is open and hence is the union of its countably many connected components, which are disjoint open intervals. In a slight abuse of language we also call $M_\gamma(\omega)$ a component of M_γ if ω is a component of $\{y > 0\}$. Thus, M_γ consists of at most countably many components, which are connected at the axis of revolution.

In the following lemma and corollary we collect some regularity properties of γ and M_γ that follow from an L^2 -bound on the second fundamental form. The focus here is on regions of $\{y > 0\}$ near the axis of revolution.

Lemma 2.1. *Let $\gamma = (x, y) : I \rightarrow \mathbb{R}^2$, $y \geq 0$ be a Lipschitz curve that satisfies $\gamma \in W_{\text{loc}}^{2,1}(\{y > 0\}; \mathbb{R}^2)$, $|\gamma'| \equiv q_\gamma > 0$ in $\{y > 0\}$, and*

$$\int_{M_\gamma(\{y > 0\})} |B|^2 d\mu < \infty.$$

Then we have $\gamma \in W_{\text{loc}}^{2,2}(\{y > 0\}; \mathbb{R}^2)$ and $y \in W^{2,1}(\{y > 0\}; \mathbb{R}^2)$. Moreover, for any connected component $\omega = (a, b)$ of $\{y > 0\}$ the curve γ belongs to $C^1(\bar{\omega}; \mathbb{R}^2)$ and has one-sided derivatives $\gamma'(a) = -\gamma'(b) = (0, |\gamma'|)$, which means that γ is perpendicular to the axis of revolution. The number of components of $\{y > 0\}$ is finite.

Proof. On any set $J \subseteq \{y > 0\}$ the y -coordinate has a positive lower bound c_J in J , thus $\gamma \in W_{\text{loc}}^{2,2}(\{y > 0\}; \mathbb{R}^2)$ follows from (2.6); using $2|K| \leq |B|^2$ and (2.5) we obtain $y \in W^{2,1}(\{y > 0\})$. The Sobolev embedding theorem then yields $x \in C_{\text{loc}}^1(\omega)$ and $y \in C^1(\bar{\omega})$ for any connected component $\omega = (a, b)$ of $\{y > 0\}$, and we aim to show that also $x \in C^1(\bar{\omega})$.

Assume for contradiction that there are sequences $t_k \rightarrow a$, $s_k \rightarrow a$ in ω such that $\lim x'(t_k) \neq \lim x'(s_k)$; if such sequences cannot be found, $x'(t)$ converges as $t \searrow a$. Since $x'(t_k)^2$ and $x'(s_k)^2$ converge to $q_\gamma^2 - y'(a)^2$, we have $\lim x'(s_k) = -\lim x'(t_k) = m \neq 0$ and $x'(t_k) < -m/2$ and $x'(s_k) > m/2$ for sufficiently large k . Thus, there is $r_k \in (t_k, s_k)$ or (s_k, t_k) such that $x'(r_k) = 0$, and from $r_k \rightarrow a$ we infer that $y'^2(a) = q_\gamma^2$. Consequently, we find $x'^2(s_k) = q_\gamma^2 - y'^2(s_k) \rightarrow 0$ and $x'^2(t_k) = q_\gamma^2 - y'^2(t_k) \rightarrow 0$ in contradiction to our assumption. Since the same argument applies at $t = b$, we obtain $x' \in C^1(\bar{\omega})$.

Next, to prove that γ is perpendicular to the axis of revolution at a , we use $y(t) \leq q_\gamma(t - a)$ in ω and the second principle curvature of M_γ to deduce that

$$\infty > \frac{q_\gamma^2}{2\pi} \int_{M(\omega)} \kappa_2^2 d\mu \geq \int_a^{a+\delta} \frac{x'^2}{t-a} dt \geq \left(\inf_{(a, a+\delta)} x'^2 \right) \int_a^{a+\delta} \frac{dt}{t-a}$$

for all $\delta \in (0, b - a)$. Continuity of x' now implies $x'(a) = 0$, and similarly we get $x'(b) = 0$. As $|\gamma'| = q_\gamma$ and $y > 0$ in ω , we find $y'(a) = -y'(b) = q_\gamma$.

Finally, by the Gauss-Bonnet formula (2.4) we have

$$\int_{M(\omega)} K d\mu = 4\pi$$

for each component ω of $\{y > 0\}$, so the number N_γ of components of $\{y > 0\}$ satisfies

$$N_\gamma \leq \frac{1}{4\pi} \sum_{\omega} \int_{M_\gamma(\omega)} |K| d\mu \leq \frac{1}{8\pi} \int_{M_\gamma(\{y > 0\})} |B|^2 d\mu$$

and is thus finite. \square

Corollary 2.2. *Let $\gamma = (x, y)$ be as in Lemma 2.1. Then M_γ has finitely many components which are connected at the axis of revolution. Each component is an immersed C^1 -surface and a $W^{2,2}$ -surface in $\{y > 0\}$, that is, away from the axis of revolution.*

Remark. The properties $y \in W^{2,1}(\{y > 0\})$ and $x \in C^1(\{y > 0\})$, but $x' \notin W^{2,1}(\{y > 0\})$ in Lemma 2.1 are sharp, as the following example shows. Let

$$\psi(t) = \frac{\sin \ln(1/t) + 1}{\ln(1/t)}$$

for $t \in (0, t_0)$ with t_0 sufficiently small that $\psi(t) \in [0, 1]$ for all $t \in (0, t_0)$ and consider

$$x'(t) = \cos(\pi/2 - \psi(t)) = \sin \psi(t), \quad y'(t) = \sin(\pi/2 - \psi(t)) = \cos \psi(t)$$

with $x(0) = y(0) = 0$. As $t \rightarrow 0$, $\psi(t)$ converges to 0 and we have $x'(t) \sim \psi(t)$, $y'(t) \sim 1$, and $y(t) \sim t$ for all small t , where $a \lesssim b$ denotes $a \leq Cb$ with a constant $C > 0$ and $a \sim b$ means $a \lesssim b \lesssim a$. Thus we obtain

$$\int \kappa_2^2 d\mu \sim \int \frac{\psi^2}{t} dt \lesssim \int \frac{dt}{t(\ln(1/t))^2} < \infty.$$

Moreover, the derivative of ψ is

$$\psi'(t) = -\frac{\cos \ln(1/t)}{t \ln(1/t)} + \frac{\sin \ln(1/t) + 1}{t(\ln(1/t))^2},$$

which implies

$$\int \kappa_1^2 d\mu \sim \int \psi'^2 t dt \lesssim \int \frac{dt}{t(\ln(1/t))^2} < \infty.$$

On the other hand, we have $x'' = \psi' \cos \psi \sim \psi'$ for small t and

$$\int |\psi'| dt \sim \int \frac{|\cos \ln(1/t)|}{t \ln(1/t)} dt = \infty,$$

thus $x \notin W^{2,1}((0, t_0))$. Furthermore, $y'' = -\psi' \sin \psi \sim -\psi' \psi$ and

$$\int |\psi' \psi|^p dt \sim \int \frac{dt}{t^p (\ln(1/t))^{2p}} = \infty$$

for any $p > 1$ yield $y'' \notin L^p((0, t_0))$. Note that M_γ is embedded due to $x' \geq 0$.

2.3 Length bound

To establish compactness of energy bounded sequences, we need bounds on the curves that are derived from bounds on the curvature integrals in the energy. Two such results, which are well-known and valid for arbitrary smoothly immersed surfaces, are [24, Lemma 1.1] and [26], which relate the extrinsic and intrinsic diameter of a surface to its mean curvature. The proof of both results hinges on the fact that in an arbitrary ball the mean curvature and the area cannot be small at the same time; the diameter bounds are then obtained by a covering argument. For closed surfaces of revolution, however, there is a straightforward proof that the mean curvature integral bounds the length of the generating curve.

Lemma 2.3. *Let $\gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) \cap W_{\text{loc}}^{2,1}(I; \mathbb{R}^2)$ be a curve such that $y(I) \subset (0, \infty)$ and $y(\partial I) = \{0\}$. Then*

$$\int_{M_\gamma} |H| d\mu \geq 2\pi \mathcal{L}_\gamma.$$

Proof. We may assume that the mean curvature integral is finite because otherwise there is nothing to prove. Without loss of generality we also assume that $\gamma: (0, \mathcal{L}_\gamma) \rightarrow \mathbb{R}^2$ is parametrised by arc length. If $x' \geq 0$ in I , there is an angle φ that is weakly differentiable in I and satisfies $\varphi \in [-\pi/2, \pi/2]$. Then we obtain

$$\begin{aligned} \int_{M_\gamma} H d\mu &= 2\pi \int_0^{\mathcal{L}_\gamma} \left(-\varphi' + \frac{\cos \varphi}{y} \right) y dt = 2\pi \int_0^{\mathcal{L}_\gamma} \varphi y' + \cos \varphi dt - 2\pi \varphi y|_0^{\mathcal{L}_\gamma} \\ &= 2\pi \int_0^{\mathcal{L}_\gamma} \varphi \sin \varphi + \cos \varphi dt \\ &\geq 2\pi \mathcal{L}_\gamma, \end{aligned}$$

because $\varphi \sin \varphi + \cos \varphi \geq 1$. In general, when $x' \geq 0$ does not hold, we consider the curve $\tilde{\gamma} = (\tilde{x}, \tilde{y})$ defined by

$$\tilde{y} = y \quad \text{and} \quad \tilde{x}(t) = x(0) + \int_0^t |x'(s)| ds.$$

We clearly have $|\tilde{\gamma}'| = |\gamma'| = 1$, and a simple calculation shows $\tilde{H} = H \operatorname{sign} x'$ almost everywhere. Therefore, we conclude

$$\int_{M_\gamma} |H| d\mu \geq \int_{M_{\tilde{\gamma}}} \tilde{H} d\tilde{\mu} \geq 2\pi \mathcal{L}_{\tilde{\gamma}} = 2\pi \mathcal{L}_\gamma. \quad \square$$

Remark. The inequality in Lemma 2.3 is actually strict: equality in the above calculation means $\varphi \sin \varphi + \cos \varphi = 1$, which holds only if $\varphi \equiv 0$ and is thus impossible for a nontrivial closed surface of revolution. Moreover, the inequality is sharp, as can be seen by a cylinder with spherical caps when the radius tends to zero.

3 Energies and Γ -convergence

3.1 Approximate setting

Recall from the introduction that we aim to approximate the energy (1.1) by

$$\mathcal{E}_\varepsilon(\gamma, u) = \int_{M_\gamma} k(u) (H - H_s(u))^2 + k_G(u) K d\mu + \int_{M_\gamma} \varepsilon |\nabla_{M_\gamma} u|^2 + \frac{1}{\varepsilon} W(u) d\mu. \quad (3.1)$$

The prescribed areas of the lipid phases translate into constraints on the area of M_γ and on the phase integral $\int_{M_\gamma} u d\mu$: if the areas of the lipid phases are given by A^+ and A^- , we require that

$$\mathcal{A}_\gamma = A_0 := A^+ + A^- \quad \text{and} \quad \int_{M_\gamma} u d\mu = m A_0, \text{ where } m = (A^+ - A^-)/A_0.$$

We assume that the double well potential $W: \mathbb{R} \rightarrow [0, \infty)$ is a continuous function that vanishes only in ± 1 and, for technical reasons, is C^2 around these points. We let $H_s: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and bounded extension of the spontaneous curvatures $H_s^\pm \in \mathbb{R}$ such that $H_s(\pm 1) = H_s^\pm$. For the bending rigidities $k^\pm > 0$ we suppose that $k: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and bounded extension of $k(\pm 1) = k^\pm$ satisfying

$$\inf_{u \in \mathbb{R}} k(u) =: k_0 > 0$$

and that $\tilde{k}_G: \mathbb{R} \rightarrow (-\infty, 0]$ is a bounded and continuous extension of $\tilde{k}_G(\pm 1) = k_G^\pm \leq 0$. Moreover, we require

$$k(u) > -\frac{\tilde{k}_G(u)}{2} \geq -\frac{k_G(u)}{2} \quad \text{uniformly in } u \in \mathbb{R}, \quad (3.2)$$

where

$$k_G(u) = \min(u^2, 1)\tilde{k}_G(u).$$

Experimental measurements of the Gauss rigidity are scarce, but available data suggest that for some membranes $-1 < k_G/(2k) < 0$ and thus (3.2) are satisfied [23, 25]. Furthermore, our assumptions are mathematically motivated by the inequalities

$$\mathcal{E}_\varepsilon(\gamma, u) \geq C \int_{M_\gamma} |H|^2 d\mu - C, \quad (3.3)$$

$$k(u) (H - H_s(u))^2 + k_G(u)K \geq -C, \quad (3.4)$$

which are necessary to obtain a suitable compactness result and the Γ -convergence lower bound; here C is a generic constant independent of γ . Indeed, expanding the quadratic term on the left hand side of (3.4) and applying Young's inequality with some $\delta > 0$ to the mixed term $2HH_s$ yields

$$\begin{aligned} & k(u) (H - H_s(u))^2 + k_G(u)K \\ &= -\frac{k_G(u)}{2}|B|^2 + \left(k(u) + \frac{k_G(u)}{2}\right) H^2 + k(u)H_s(u)^2 - 2k(u)HH_s(u) \\ &\geq -\frac{k_G(u)}{2}|B|^2 + \left(k(u)(1 - \delta) + \frac{k_G(u)}{2}\right) H^2 - k(u)\frac{1 - \delta}{\delta}H_s(u)^2, \end{aligned}$$

hence, (3.4) is satisfied, provided that $k_G(u) \leq 0$ and (3.2) hold. Then (3.3) is true, if additionally the area of M_γ is prescribed.

Most interesting is the factor u^2 in our definition of k_G , which differs from other diffuse models for the lipid phases [11, 28] where the extended Gauss rigidities are bounded away from 0. The latter studies do not consider topological changes in the limit, which, however, is necessary to establish an equi-coercivity result. The purpose of the u^2 is to allow the construction of appropriate recovery sequences; see the end of Section 4.3.3 for the details.

We study (3.1) for membranes $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, where

$$\begin{aligned} \mathcal{C}_\varepsilon := & \left\{ \gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) \cap W_{\text{loc}}^{2,1}(I; \mathbb{R}^2) : \right. \\ & \left. |\gamma'| = \text{const}, y(\partial I) = \{0\}, y(I) \subset (0, \infty), \int_{M_\gamma} |B|^2 d\mu < \infty, \mathcal{A}_\gamma = A_0 \right\} \end{aligned}$$

and

$$\mathcal{P}_\varepsilon := \left\{ u \in W_{\text{loc}}^{1,1}(I) : \int_{M_\gamma} |\nabla_{M_\gamma} u|^2 d\mu < \infty, \|u\|_\infty \leq C_0, \int_{M_\gamma} u d\mu = mA_0 \right\}.$$

The first three conditions in the definition of \mathcal{C}_ε ensure that γ is parametrised with constant speed and that M_γ is a closed surface. The L^2 -bound on the second fundamental form of M_γ together with the first two conditions on the phase fields ensure that the energy (3.1) is well-defined on $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. The requirement $\|u\|_\infty \leq C_0$ with a large constant $C_0 \gg 1$ seems rather strong, but phase fields with small energy are expected to be close to the interval $[-1, 1]$ anyway. In fact, in many places in the proof the L^∞ -bound can be replaced by a less restrictive condition; compare [18, Proposition 3].

Although the set \mathcal{P}_ε depends on the chosen $\gamma \in \mathcal{C}_\varepsilon$ via the phase area constraint, we suppress this fact in the notation, because we usually consider pairs or membranes (γ, u) . Instead, we highlight the affiliation to the approximate energy by the index ε in $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$. In the following, we write M_ε instead of M_{γ_ε} and so forth, when considering sequences (γ_ε) of curves. If necessary or useful for clarification we add the curve or ε as index to other quantities such as H_γ , μ_γ , or y_γ .

The energy (3.1) is invariant under reparametrisations that preserve the orientation and the regularity properties of γ . In particular, if (γ, u) satisfies all requirements of $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ but only $|\gamma'| \neq 0$ instead of $|\gamma'| = \text{const}$, the corresponding constant speed parametrisation belongs to $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ and has the same energy. Hence, considering only $|\gamma'| = \text{const}$ is no geometric restriction.

3.2 Limit setting

Our limit energy is

$$\mathcal{E}(\gamma, u) = \int_{M_\gamma} k(u)(H - H_s(u))^2 + k_G(u)K d\mu + \sigma \mathcal{H}^1(M_\gamma(S_u))$$

for curves with parametrisations γ in

$$\begin{aligned} \mathcal{C} := \Big\{ \gamma = (x, y) \in C^{0,1}(I; \mathbb{R}^2) \cap W_{\text{loc}}^{2,1}(\{y > 0\}; \mathbb{R}^2) : \\ |\gamma'| = \text{const}, y(\partial I) = \{0\}, y \geq 0, \mathcal{H}^0(\{y = 0\}) < \infty, \\ \int_{M(\{y > 0\})} |B|^2 d\mu < \infty, \mathcal{A}_\gamma = A_0 \Big\} \end{aligned}$$

and associated phase fields u in

$$\mathcal{P} := \left\{ u: I \rightarrow \{\pm 1\} \text{ piecewise constant} : \int_M u d\mu = mA_0, \mathcal{H}^1(M_\gamma(S_u)) < \infty \right\}.$$

Here $S_u \subset \{y > 0\}$ denotes the countable jump set of u in $\{y > 0\}$, and we call $s \in S_u$ and the corresponding circle $M_\gamma(\{s\})$ an interface of (γ, u) . The constant σ is given by

$$\sigma = 2 \int_{-1}^1 \sqrt{W(u)} du,$$

and

$$\mathcal{H}^1(M_\gamma(S_u)) = 2\pi \sum_{s \in S_u} y(s)$$

is the one-dimensional Hausdorff measure of the union of the countably many circles $M_\gamma(S_u)$.

The difference between \mathcal{C}_ε and \mathcal{C} is that $\gamma \in \mathcal{C}$ may touch the axis of revolution also in the interior of I , but this can happen only at finitely many points. For $\gamma \in \mathcal{C}$ we infer from Lemma 2.1 and the subsequent corollary that M_γ consists of finitely many components which are C^1 -surfaces and $W^{2,2}$ -surfaces away from the axis of revolution.

The set \mathcal{P} resembles the set of special functions of bounded variation SBV with values in $\{\pm 1\}$, weighted with the height y of the generating curve $\gamma \in \mathcal{C}$. Indeed, for $u \in \mathcal{P}$ and any $J \Subset \{y > 0\}$ we have $u \in SBV(J; \{\pm 1\})$, but as jumps of height 2 may accumulate near the axis of revolution, $u \notin SBV(I)$ in general. Points in $\{y = 0\}$ can be jump points of u or singular points where one or both one-sided limits are undefined. We emphasise that in our notation S_u only contains points in $\{y > 0\}$, because the restriction of u to $\{y = 0\}$ does not contribute to the energy \mathcal{E} .

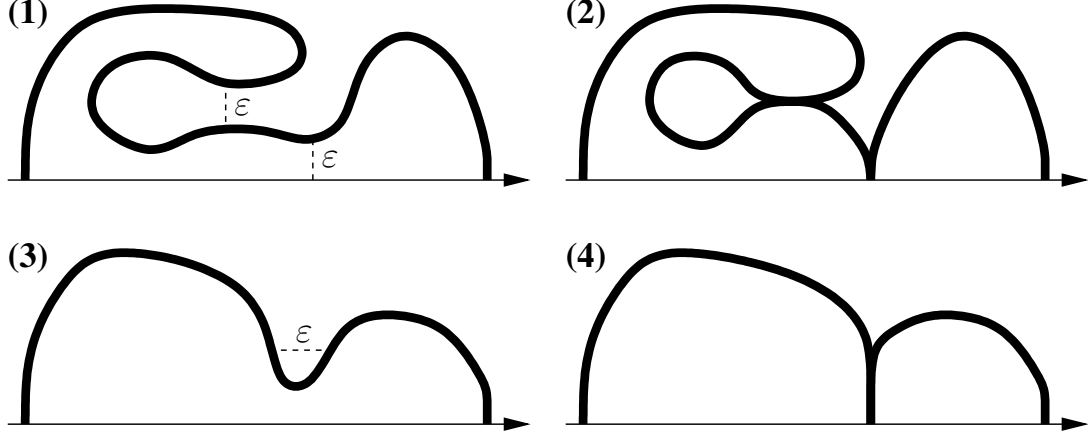


Figure 3.1: Examples of embedded curves (1) and (3) in \mathcal{C}_ε that lead to non-embedded limits (2) and (4), respectively, in \mathcal{C} . The curves in (3) and (4) satisfy even the stronger condition $x' \geq 0$ which prevents a component from touching itself, but different components may still touch each other in vertical segments near the axis of revolution.

3.3 Γ -convergence

We extend \mathcal{E}_ε and \mathcal{E} to $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ by setting $\mathcal{E}_\varepsilon(\gamma, u) = \mathcal{E}(\gamma, u) = \infty$ whenever (γ, u) does not belong to $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ and $\mathcal{C} \times \mathcal{P}$, respectively. Our approximation theorem is the following.

Theorem 3.1. *The energies \mathcal{E}_ε are equi-coercive, that is, any sequence $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ with uniformly bounded energy admits a subsequence that converges strongly in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ to some $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$. Furthermore, \mathcal{E}_ε Γ -converges to \mathcal{E} as $\varepsilon \rightarrow 0$, that is,*

- for any sequence $(\gamma_\varepsilon, u_\varepsilon)$ that converges to some (γ, u) in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ we have

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{E}(\gamma, u);$$

- for any (γ, u) with finite energy $\mathcal{E}(\gamma, u)$ there is a recovery sequence $(\gamma_\varepsilon, u_\varepsilon)$ that converges to (γ, u) in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$ and satisfies

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \leq \mathcal{E}(\gamma, u).$$

Remark (Existence of minimisers). The energy \mathcal{E}_ε is bounded from below on $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$; thus, there is a sequence $(\gamma_\varepsilon, u_\varepsilon)$ such that $\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) = \inf \mathcal{E}_\varepsilon + o(1)_{\varepsilon \rightarrow 0}$. From equi-coercivity and Γ -convergence we infer that a subsequence of $(\gamma_\varepsilon, u_\varepsilon)$ converges to a minimiser of \mathcal{E} in $\mathcal{C} \times \mathcal{P}$, whose existence is thus established; see for instance [3] for the details.

Remark (Embeddedness). Our setting and result, which are entirely based on parametrisations, do neither need nor guarantee embeddedness. Even if \mathcal{E}_ε is considered only on the subset of embedded membranes or for curves $\gamma = (x, y)$ that satisfy the stronger “generalised graph” condition $x' \geq 0$, which is preserved under our convergence, limit curves can touch themselves; see Figure 3.1 for two examples.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 is divided into the three steps equi-coercivity, lower bound and upper bound inequality.

4.1 Equi-coercivity

Recalling (3.3), which states

$$\mathcal{E}_\varepsilon(\gamma, u) \geq C_1 \int_{M_\gamma} H^2 d\mu - C$$

for some constants $C_1, C > 0$ independent of $(\gamma, u) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, and adding $-8\pi C_1 = -C_1 \int_{M_\gamma} 2K d\mu$ to both sides, we find

$$\mathcal{E}_\varepsilon(\gamma, u) + C \geq C_1 \int_{M_\gamma} |B|^2 d\mu \geq C_1 \int_{M_\gamma} |K| d\mu.$$

This means, that $\mathcal{E}_\varepsilon(\gamma, u)$ bounds the L^2 -norms of B and H as well as the L^1 -norm of the Gauss curvature of M_γ . Since moreover $\mathcal{E}_\varepsilon(\gamma, u) + C$ also bounds the phase field energy from above, compactness for curves and phase fields can be established separately.

Lemma 4.1. *Let $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ be a sequence with uniformly bounded energy $\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon)$. Then there are $\gamma = (x, y) \in \mathcal{C}$ and a subsequence, not relabelled, such that*

- $\gamma_\varepsilon \xrightarrow{*} \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2)$;
- $\gamma_\varepsilon \rightharpoonup \gamma$ in $W_{\text{loc}}^{2,2}(\{y > 0\}; \mathbb{R}^2)$; and
- $\gamma_\varepsilon \rightarrow \gamma$ in $W^{1,p}(I; \mathbb{R}^2)$ for any $p \in [1, \infty)$.

Proof. Let $\gamma_\varepsilon = (x_\varepsilon, y_\varepsilon)$ and $|\gamma'_\varepsilon| = q_\varepsilon$. Using Lemma 2.3 and Hölder's inequality we find

$$2\pi q_\varepsilon |I| = 2\pi \mathcal{L}_\varepsilon \leq \int_{M_\varepsilon} |H_\varepsilon| d\mu_\varepsilon \leq \left(\mathcal{A}_\varepsilon \int_{M_\varepsilon} H_\varepsilon^2 d\mu_\varepsilon \right)^{1/2},$$

which bounds the sequence (q_ε) from above. Furthermore, translations in x -direction do not change the energy, so we may assume that all γ_ε have a common end point and conclude that (γ_ε) is bounded in $W^{1,\infty}(I; \mathbb{R}^2)$. We can therefore extract a subsequence such that $q_\varepsilon \rightarrow q$ in \mathbb{R} and $\gamma_\varepsilon \xrightarrow{*} \gamma$ in $W^{1,\infty}(I; \mathbb{R}^2) = C^{0,1}(I; \mathbb{R}^2)$; by compact embedding of $W^{1,\infty}$ into C^0 , the convergence of γ_ε is uniform in \bar{I} . This clearly implies $y \geq 0$ and $y(\partial I) = \{0\}$, but also $q > 0$ and $y \not\equiv 0$ because

$$A_0 = \mathcal{A}_\varepsilon = 2\pi q_\varepsilon \int_I y_\varepsilon dt \rightarrow 2\pi q \int_I y dt.$$

Without loss of generality we assume $q = 1$, thus $|\gamma'| \leq 1$ almost everywhere in I .

Taking into account only the just selected subsequence, let ε be sufficiently small so that $q_\varepsilon \leq 2$, and let $J \Subset \{y > 0\}$ and $c_J > 0$ be such that $y \geq 2c_J$ in J . By uniform convergence of y_ε we have $y_\varepsilon \geq c_J$ for all small ε , and (2.6) yields

$$\frac{1}{2\pi} \int_{M_\varepsilon} |B_\varepsilon|^2 d\mu_\varepsilon \geq \frac{1}{2\pi} \int_{M_\varepsilon(J)} \kappa_{1,\varepsilon}^2 d\mu_\varepsilon \geq \frac{c_J}{8} \int_J |\gamma''_\varepsilon|^2 dt. \quad (4.1)$$

Since the left hand side of (4.1) is uniformly bounded, a subsequence of (γ''_ε) converges weakly in $L^2(J; \mathbb{R}^2)$ to some γ''_J . For this subsequence, γ_ε converges weakly in $W^{2,2}(J; \mathbb{R}^2)$, and from uniqueness of the weak limit we infer that γ''_J is the weak derivative of γ' in J and that the whole sequence converges. This proves $\gamma_\varepsilon \rightharpoonup \gamma$ in $W_{\text{loc}}^{2,2}(\{y > 0\}; \mathbb{R}^2)$, and we obtain

$$\int_{M_\gamma(J)} |B|^2 d\mu \leq \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(J)} |B_\varepsilon|^2 d\mu_\varepsilon \leq \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} |B_\varepsilon|^2 d\mu_\varepsilon \quad (4.2)$$

for any $J \in \{y > 0\}$. Exhausting $\{y > 0\}$ by $J \in \{y > 0\}$, we conclude

$$\int_{M_\gamma(\{y>0\})} |B|^2 d\mu < \infty,$$

because the right hand side of (4.2) is finite and independent of J .

From the compact embedding of $W^{2,2}$ into C^1 we know that γ_ε converges strongly to γ in $C^1_{\text{loc}}(\{y > 0\}; \mathbb{R}^2)$, which implies $\gamma'_\varepsilon \rightarrow \gamma'$ pointwise in $\{y > 0\}$. Thus, we find $|\gamma'| = \lim |\gamma'_\varepsilon| = \lim q_\varepsilon = 1$ in $\{y > 0\}$ and

$$\mathcal{A}_\gamma = 2\pi \int_I |\gamma'| y dt = 2\pi \int_{\{y>0\}} y dt = \lim_{\varepsilon \rightarrow 0} 2\pi \int_{\{y>0\}} q_\varepsilon y_\varepsilon dt = \lim_{\varepsilon \rightarrow 0} \mathcal{A}_\varepsilon = A_0.$$

Finally, to conclude $\gamma \in \mathcal{C}$ we have to show that $\{y = 0\}$ is finite. This also yields strong convergence in $W^{1,p}(I; \mathbb{R}^2)$, because it implies $\gamma'_\varepsilon \rightarrow \gamma'$ almost everywhere in I . Assume for contradiction that J is a non-empty open subset of $\{y = 0\}$. From

$$\left(\int_J |x'_\varepsilon| dt \right)^2 = \left(\int_J \frac{|x'_\varepsilon|}{\sqrt{q_\varepsilon y_\varepsilon}} \sqrt{q_\varepsilon y_\varepsilon} dt \right)^2 \leq \frac{\mathcal{A}_\varepsilon(J)}{4\pi^2} \int_{M_\varepsilon} \kappa_{2,\varepsilon}^2 d\mu_\varepsilon$$

we then see that $x'_\varepsilon \rightarrow 0$ and $y_\varepsilon'^2 = q_\varepsilon^2 - x_\varepsilon'^2 \rightarrow 1$ in $L^1(J)$, which contradicts $y' = 0$ almost everywhere in $\{y = 0\}$. Consequently, $\{y = 0\}$ does not contain interior points, and since by Lemma 2.1 the number of components of $\{y > 0\}$ is finite, we conclude $\mathcal{H}^0(\{y = 0\}) < \infty$. \square

Lemma 4.2. *Let $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ and $\gamma \in \mathcal{C}$ be as in Lemma 4.1. Then there exist a countable set $S \subset I$ with $S \cap J$ finite for any $J \in \{y > 0\}$ and $u \in \mathcal{P}$ with $S_u \subset S$ such that for a subsequence $u_\varepsilon \rightarrow u$ in measure, almost everywhere in I , and in $L^p(I)$ for $p \in [1, \infty)$.*

Proof. We restrict ourselves to a subsequence of γ_ε that converges to γ according to Lemma 4.1; as above, we let $|\gamma'_\varepsilon| \equiv q_\varepsilon$ and without loss of generality $|\gamma'| \equiv 1$. Uniform convergence implies that for $J \in \{y > 0\}$ there is $c_J > 0$ such that $y_\varepsilon \geq c_J$ in J for all sufficiently small ε . Therefore, we have

$$\frac{1}{2\pi} \int_{M_\varepsilon(J)} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) d\mu_\varepsilon \geq c_J \int_J \frac{\varepsilon}{q_\varepsilon} |u'_\varepsilon|^2 + \frac{q_\varepsilon}{\varepsilon} W(u_\varepsilon) dt \quad (4.3)$$

and the well-known arguments of Modica and Mortola [18, 19] apply in J ; see also [3, Lemma 6.2 and Remark 6.3] for a proof in one dimension. The outcome is a finite set of points $S_J \subset J$ and a piecewise constant function $u: J \rightarrow \{\pm 1\}$ whose jump set is contained in S_J such that a subsequence of u_ε converges to u in measure and almost everywhere in $J \setminus S$. Since (u_ε) is uniformly bounded in $L^\infty(I)$, convergence in $L^p(I)$ for any $p < \infty$ follows.

Exhausting $\{y > 0\}$ by a sequence of increasing sets such as $J_k = \{y > 1/k\}$ for $k \rightarrow \infty$ and taking a diagonal sequence, we find an at most countable set $S \subset \{y > 0\}$ and a function $u: \{y > 0\} \rightarrow \{\pm 1\}$ whose jump set is contained in S . Moreover, a subsequence of (u_ε) converges to u in measure and almost everywhere in $\{y > 0\}$. Then $\mathcal{H}^0(\{y = 0\}) < \infty$ and $\|u_\varepsilon\|_\infty \leq C_0$ provide convergence in $L^p(I)$ for any $1 \leq p < \infty$, and taking convergence of y_ε and $|\gamma'_\varepsilon|$ into account, we obtain

$$mA_0 = \int_{M_\varepsilon} u_\varepsilon d\mu_\varepsilon \rightarrow \int_{M_\gamma} u d\mu$$

as $\varepsilon \rightarrow 0$. The bound $\mathcal{H}^1(M_\gamma(S_u)) < \infty$ follows from (4.3) and Young's inequality; the details are given in the lower bound section and are thus here omitted. \square

Remark. In the classical one-dimensional setting without the area measure, a uniform L^∞ -bound for the phase fields is in fact a result of the uniform energy bound; see [3]. In our case, however, this bound depends in $J \in \{y > 0\}$ on the constant c_J , which is essentially the infimum of y on J , and tends to infinity as $c_J \rightarrow 0$.

4.2 Lower bound

Next we prove the lower bound inequality

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \geq \mathcal{E}(\gamma, u) \quad (4.4)$$

whenever $(\gamma_\varepsilon, u_\varepsilon)$ converges to (γ, u) in $W^{1,1}(I; \mathbb{R}^2) \times L^1(I)$. It suffices to examine the case when the left hand side of (4.4) is finite and to consider a subsequence such that the lower limit is attained. Then by definition $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$, and our compactness argument yields $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ and the convergence properties listed in Lemmas 4.1 and 4.2.

Recalling the formulas $\kappa_{1,\varepsilon} = -\gamma_\varepsilon'' \cdot \gamma_\varepsilon'^\perp / q_\varepsilon^3$ and $\kappa_{2,\varepsilon} = x_\varepsilon' / (q_\varepsilon y_\varepsilon)$ for the principal curvatures, we find that $\gamma_\varepsilon \rightarrow \gamma$ in $W_{\text{loc}}^{2,2}(\{y > 0\}; \mathbb{R}^2)$ implies weak convergence of H_ε and K_ε in $L_{\text{loc}}^2(\{y > 0\})$. Together with $y_\varepsilon q_\varepsilon \rightarrow yq$ uniformly, $u_\varepsilon \rightarrow u$ in $L^1(I)$, and the L^∞ -bounds for k and k_G this yields

$$\begin{aligned} \int_{M_\gamma(J)} k(u)(H - H_s(u))^2 + k_G(u)K \, d\mu \\ \leq \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(J)} k(u_\varepsilon)(H_\varepsilon - H_s(u_\varepsilon))^2 + k_G(u_\varepsilon)K_\varepsilon \, d\mu_\varepsilon \end{aligned}$$

for any $J \in \{y > 0\}$. Adding temporarily CA_0 , where C is the constant in (3.4), to make the integral on the right hand side non-negative, we estimate the latter by extending it to the whole surface M_ε and exhaust $\{y > 0\}$ by intervals $J \in \{y > 0\}$ on the left hand side. Thereby, we obtain the bulk lower bound

$$\begin{aligned} \int_{M_\gamma(\{y>0\})} k(u)(H - H_s(u))^2 + k_G(u)K \, d\mu \\ \leq \liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} k(u_\varepsilon)(H_\varepsilon - H_s(u_\varepsilon))^2 + k_G(u_\varepsilon)K_\varepsilon \, d\mu_\varepsilon. \end{aligned} \quad (4.5)$$

To analyse the interface energy let $s \in S_u$ and fix an interval $J \in \{y > 0\}$ such that $\bar{J} \cap S_u = \{s\}$, which exists because $S_u \cap \{y > y(s)/2\}$ is finite. From the convergence of u_ε we deduce that there are points $a_\varepsilon, b_\varepsilon \in J$ with $a_\varepsilon < s < b_\varepsilon$ or $b_\varepsilon < s < a_\varepsilon$ such that $a_\varepsilon \rightarrow s$, $b_\varepsilon \rightarrow s$, $u_\varepsilon(a_\varepsilon) \rightarrow -1$, and $u_\varepsilon(b_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Assuming without loss of generality that $a_\varepsilon < b_\varepsilon$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{M_\varepsilon(a_\varepsilon, b_\varepsilon)} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon &\geq \left(\inf_{(a_\varepsilon, b_\varepsilon)} y_\varepsilon \right) \int_{a_\varepsilon}^{b_\varepsilon} 2\sqrt{W(u_\varepsilon)} |u_\varepsilon'| \, dt \\ &\geq \left(\inf_{(a_\varepsilon, b_\varepsilon)} y_\varepsilon \right) \left| \int_{u_\varepsilon(a_\varepsilon)}^{u_\varepsilon(b_\varepsilon)} 2\sqrt{W(u)} \, du \right| \end{aligned}$$

thanks to Young's inequality and a change of variables. Taking the lower limit yields

$$\liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon(a_\varepsilon, b_\varepsilon)} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \, d\mu_\varepsilon \geq 2\pi y(s) \int_{-1}^1 2\sqrt{W(u)} \, du = 2\pi y(s)\sigma. \quad (4.6)$$

The above argument applies to each point of any finite subset S of S_u , and in addition we may extend the integral on the left hand side of (4.6) to the whole surface to obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) d\mu_\varepsilon \geq \sigma \mathcal{H}^1(M_\gamma(S)).$$

Since the left hand side is independent of S , the interface lower bound inequality

$$\liminf_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} \varepsilon |\nabla_{M_\varepsilon} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) d\mu_\varepsilon \geq \sigma \mathcal{H}^1(M_\gamma(S_u)) \quad (4.7)$$

follows from taking the supremum over all finite sets $S \subset S_u$. Combining (4.7) and (4.5) yields the lower bound inequality (4.4).

4.3 Upper bound

We now construct a recovery sequence for (γ, u) with finite energy $\mathcal{E}(\gamma, u)$. To this end, we first show that (γ, u) can be approximated by membranes with finitely many interfaces. For such a membrane, we then obtain a recovery sequence by changing the curve essentially only near component boundaries and the phase field only around interfaces and component boundaries. Finally, a diagonal sequence recovers (γ, u) .

Throughout this section we assume without loss of generality that $|\gamma'| \equiv 1$.

4.3.1 Approximation by finite number of interfaces

Lemma 4.3. *Assume that $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ has countably many interfaces. Then there exists $(\gamma, u_\delta) \in \mathcal{C} \times \mathcal{P}$ for sufficiently small $\delta > 0$, each with a finite number of interfaces, such that $u_\delta \rightarrow u$ in $L^p(I)$ for any $p \in [1, \infty)$ and $\mathcal{E}(\gamma, u_\delta) \rightarrow \mathcal{E}(\gamma, u)$ as $\delta \rightarrow 0$.*

Proof. Let $\gamma = (x, y)$ and 3δ be smaller than the minimal length of a component of $\{y > 0\}$. We construct u_δ by omitting interfaces whose distance on γ to a component boundary is less than δ . More precisely, for a component $\omega = (a, b)$ of $\{y > 0\}$ we let $a_\delta = a + \delta$ and $b_\delta = b - \delta$, which implies $a_\delta < b_\delta$ and $\mathcal{L}_\gamma(a, a_\delta) = \mathcal{L}_\gamma(b_\delta, b) = \delta$, and define u_δ on ω to be the continuous extension of u from (a_δ, b_δ) to ω , that is,

$$u_\delta = \begin{cases} u & \text{in } (a_\delta, b_\delta), \\ \lim_{t \searrow a_\delta} u(t) & \text{in } (a, a_\delta], \\ \lim_{t \nearrow b_\delta} u(t) & \text{in } [b_\delta, b). \end{cases}$$

Since the number of components N_γ is finite, this can be done separately for each component, and the composition yields a membrane (γ, u_δ) with finitely many interfaces. By construction, we have $|u - u_\delta| \leq 2$ and $y \leq \delta$ in $(a, a_\delta) \cup (b_\delta, b)$, so we find

$$\int_{M_\gamma} |u - u_\delta|^p d\mu \leq 2^{p+1} N_\gamma \delta^2 \quad (4.8)$$

and $u_\delta \rightarrow u$ as $\delta \rightarrow 0$ in $L^p(I)$ for any $p \in [1, \infty)$. Furthermore,

$$|\mathcal{H}^1(M_\gamma(S_u)) - \mathcal{H}^1(M_\gamma(S_{u_\delta}))| \leq \mathcal{H}^1(M(S_u \cap \{y \leq \delta\}))$$

as well as

$$\begin{aligned} & \left| \int_{M_\gamma} k(u)(H - H_s(u))^2 + k_G(u)K - k(u_\delta)(H - H_s(u_\delta))^2 - k_G(u_\delta)K d\mu \right| \\ & \leq \int_{M_\gamma(\{y \leq \delta\})} 2\|k\|_\infty H^2 + 4\|kH_s\|_\infty (|H| + \|H_s\|_\infty) + 2\|k_G\|_\infty |K| d\mu \end{aligned}$$

vanish in the limit $\delta \rightarrow 0$, and we deduce $\mathcal{E}(\gamma, u_\delta) \rightarrow \mathcal{E}(\gamma, u)$.

Finally, for sufficiently small δ there is an interface $s \in S_u \cap S_{u_\delta}$ that is independent of δ and whose distance to all other interfaces is greater than δ . According to (4.8) the error in the phase constraint is at most of order δ^2 , so it suffices to move s by an order of at most δ^2 to the left or right to recover the integral constraint $\int_M u_\delta d\mu = mA_0$. This additional change yields $u_\delta \in \mathcal{P}$ and does not disturb the convergence of phase fields and energy. \square

In virtue of Lemma 4.3 we assume from now on that (γ, u) has only finitely many interfaces. Then u is either continuous at points in $\{y = 0\}$ or has a well-defined jump. Moreover, the minimal distance between two interfaces and from an interface to the boundary of its component is positive. Hence, for any interface $s \in S_u$ there is an interval $J \Subset \{y > 0\}$ that contains s but no other interface, and for any component boundary point $s \in \{y = 0\} \setminus \partial I$ there is an interval $J \subset I$ that contains s but no other component boundary or interface.

4.3.2 Local interface recovery

The recovery of a phase field u with finitely many jumps follows the lines of the Modica-Mortola theory for phase transitions. The main difference is the inhomogeneity due to the area measure $d\mu = 2\pi y dt$, but since u will be changed only in an interval of order $\sqrt{\varepsilon}$ around each interface, this issue is easily dealt with.

It is well known, see for instance [1], that in the classical one-dimensional setting the ε -energy-minimal profile for a transition of u_ε from -1 to $+1$ is obtained by minimising

$$G_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon |u'|^2 + \frac{1}{\varepsilon} W(u) dt$$

among functions u that satisfy $u(0) = 0$ and $u(\pm\infty) = \pm 1$. Indeed, setting $u_\varepsilon(t) = u(t/\varepsilon)$ we observe

$$G_\varepsilon(u_\varepsilon) = G_1(u) \geq 2 \int_{\mathbb{R}} \sqrt{W(u)} u' dt = 2 \int_{\mathbb{R}} \sqrt{W(u)} du = \sigma,$$

and equality holds if and only if

$$u' = \sqrt{W(u)}. \quad (4.9)$$

Equation (4.9) admits a local solution p with initial condition $p(0) = 0$, because $\sqrt{W(\cdot)}$ is continuous. Since the constants $+1$ and -1 are a global super- and sub-solution of (4.9), p can be extended to the whole real line, and due to $W(p) > 0$ for $p \in (-1, +1)$, we obtain $p(t) \rightarrow \pm 1$ as $t \rightarrow \pm\infty$. As a consequence, $p(t/\varepsilon)$ is admissible and minimises G_ε . Furthermore, by symmetry of W we can presume $-p(-t) = p(t)$ and need to know the profile only for $t \geq 0$.

Let $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ have finitely many interfaces and consider $s \in S_u$ and $J \Subset \{y > 0\}$ such that $\bar{J} \cap S_u = \{s\}$. For simplicity of notation we assume $s = 0$. Using an appropriately scaled version of the optimal profile p and a linear interpolation, we aim to construct the recovery sequence by replacing $u = \text{sign } t$ on J with

$$p_\varepsilon(t) = \begin{cases} p(t/\varepsilon) & \text{if } 0 \leq t < \sqrt{\varepsilon}, \\ p(1/\sqrt{\varepsilon}) + \frac{1}{\varepsilon}(t - \sqrt{\varepsilon}) & \text{if } \sqrt{\varepsilon} \leq t < \sqrt{\varepsilon} + \varepsilon(1 - p(1/\sqrt{\varepsilon})), \\ 1 & \text{if } \sqrt{\varepsilon} + \varepsilon(1 - p(1/\sqrt{\varepsilon})) \leq t \end{cases}$$

for $t \geq 0$ and $p_\varepsilon(t) = -p_\varepsilon(-t)$ for $t < 0$; if $u = -\text{sign } t$ in J , we use $-p_\varepsilon$. Since γ is in general not symmetric around $s = 0$ we have to correct p_ε in order to conserve the phase integral constraint.

Lemma 4.4. *There is $u_\varepsilon \in W^{1,2}(J)$ with $\{u_\varepsilon \neq u\} \Subset J$ such that $\|u_\varepsilon\|_\infty \leq C_0$, $u_\varepsilon \rightarrow u$ in $L^1(J)$, $\int_{M(J)} u_\varepsilon d\mu = \int_{M(J)} u d\mu$, and*

$$\limsup_{\varepsilon \rightarrow 0} \int_{M_\gamma(J)} \varepsilon |\nabla_{M_\gamma} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) d\mu \leq 2\pi\sigma y(s). \quad (4.10)$$

Proof. Convergence $p_\varepsilon \rightarrow u$ in $L^1(J)$ is obvious from the definition of p_ε , and the estimate (4.10) with p_ε instead of u_ε follows by taking the upper limit $\varepsilon \rightarrow 0$ in

$$\begin{aligned} \int_{M_\gamma(J)} \varepsilon |\nabla_{M_\gamma} p_\varepsilon|^2 + \frac{1}{\varepsilon} W(p_\varepsilon) d\mu &\leq 2\pi \left(\sup_{[-\varepsilon, \varepsilon]} y \right) \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} |p'(t)|^2 + W(p(t)) dt \\ &\quad + 2\pi (1 - p(1/\sqrt{\varepsilon})) \left(\sup_J y \right) \left(1 + \sup_{[-1,1]} W \right). \end{aligned}$$

To recover the constraint, let $f: J \rightarrow \mathbb{R}$ be smooth, have compact support in $J \cap \{t > 0\}$ and satisfy $\int_{M_\gamma(J)} f d\mu = 1$. Then the phase integral is conserved by $u_\varepsilon = p_\varepsilon + \alpha_\varepsilon f$ if

$$\alpha_\varepsilon = \int_{M_\gamma(J)} u - p_\varepsilon d\mu.$$

From

$$\int_{M_\gamma(0, \sqrt{\varepsilon})} 1 - p_\varepsilon d\mu \leq 2\pi \|y\|_\infty \sqrt{\varepsilon} \int_0^1 1 - p(t/\sqrt{\varepsilon}) dt = o(\sqrt{\varepsilon}),$$

we infer that α_ε is of order $o(\sqrt{\varepsilon})$, which is sufficient to ensure convergence $u_\varepsilon \rightarrow u$ in $L^1(J)$ and the energy inequality

$$\limsup_{\varepsilon \rightarrow 0} \int_{M_\gamma(J)} \varepsilon |\nabla_{M_\gamma} u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) d\mu \leq 2\pi\sigma y(s)$$

thanks to

$$\frac{1}{\varepsilon} W(\pm 1 + \alpha_\varepsilon f) = \frac{1}{\varepsilon} (W(\pm 1) + \alpha_\varepsilon f W'(\pm 1) + O(\alpha_\varepsilon^2)) = o(1).$$

By construction, we have $u_\varepsilon \in W^{1,2}(J)$ and $\|u_\varepsilon\|_\infty \leq \|p_\varepsilon\|_\infty + |\alpha_\varepsilon| \|f\|_\infty \leq C_0$ for all sufficiently small $\varepsilon > 0$. \square

Remark. Lemma 4.4 remains true if γ is replaced by a sequence γ_ε that satisfies $|\gamma'_\varepsilon| \equiv q_\varepsilon \rightarrow 1$, $|\mathcal{A}_\varepsilon(J) - \mathcal{A}_\gamma(J)| = o(\sqrt{\varepsilon})$ and $\gamma_\varepsilon \rightarrow \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for some $p \in [1, \infty)$.

4.3.3 Curve approximation and recovery

To obtain a recovery sequence for the curves we have to change segments of $\gamma = (x, y)$ near interior points on the axis of revolution. Close to the axis the second principal curvature becomes unbounded unless $x' = 0$, therefore we base our construction on scaled catenoids in order to control the mean curvature integral in the energy \mathcal{E}_ε . With the topological changes introduced by this construction and their effect on the Gauss curvature integral we deal later by adapting the phase field.

One issue with the above idea is that the catenoids have to make a C^1 -connection with the original surface, which even after taking symmetries into account can display several types of behaviour. For instance, the generating curve might leave the axis of revolution turning only in one direction, zig-zagging in x -direction, or as a vertical line segment; see Figure 4.1. In a first step we therefore reduce the number of possible situations by showing that a membrane can be approximated by membranes that only have vertical line segments near the axis of revolution.

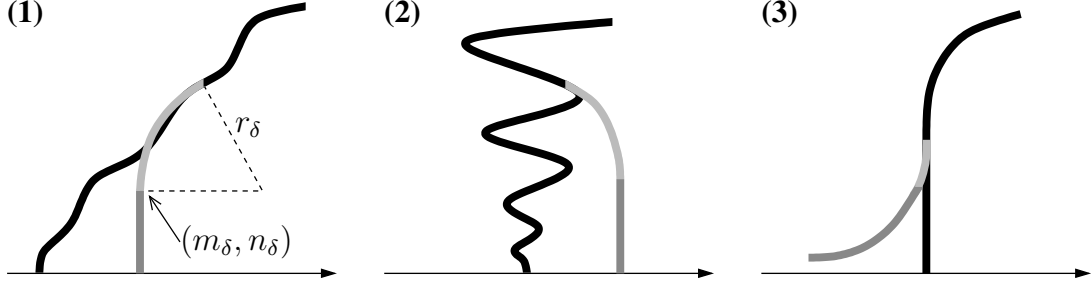


Figure 4.1: Examples of curves γ (black) near the axis of revolution and their recovery (grey): γ turns in one-direction (1), zig-zags (2), or is vertical near the axis of revolution (3); it is approximated in (1) and (2) by a circle segment (light grey) and a vertical line (dark grey) as constructed in Lemma 4.5; the vertical line in (3) is recovered by a catenary (dark grey) and a circle segment (light grey) as in Lemma 4.6.

Lemma 4.5. *Let $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ have finitely many interfaces. Then for sufficiently small $\delta > 0$ there is $(\gamma_\delta, u_\delta) \in \mathcal{C} \times \mathcal{P}$ with finitely many interfaces such that $\gamma_\delta \rightarrow \gamma$ in $W^{1,p}(I; \mathbb{R}^2)$, $u_\delta \rightarrow u$ in $L^p(I)$ for any $p \in [1, \infty)$, and $\mathcal{E}(\gamma_\delta, u_\delta) \rightarrow \mathcal{E}(\gamma, u)$ as $\delta \rightarrow 0$. Moreover, each γ_δ meets the axis of revolution in vertical line segments, that is, for any $s \in \{y = 0\}$ there are $a_\delta, b_\delta \in I$, $b_\delta < s < a_\delta$ such that y_δ restricted to (b_δ, s) and (s, a_δ) , respectively, is a vertical line.*

Proof. For the local construction around a point on the axis of revolution we consider the left boundary of one component $\omega = (a, b)$ of M_γ where γ is not vertical; at right component boundaries a mirrored version applies. For simplicity of notation we assume $a = 0$, $\gamma(0) = (0, 0)$, and $|\gamma'| = 1$ in I .

From Lemma 2.1 we know that $x'(0) = 0$ and $|y'(t)| \rightarrow 1$ as $t \searrow 0$. It is sufficient to consider the case $y'(t) \rightarrow +1$ as $t \searrow 0$, as the construction for $y'(t) \rightarrow -1$ is obtained by traversing the former one backwards. Let $J = (0, t_0) \subset \omega$ be an interval that contains no interface of (γ, u) and such that $y' \geq 1/2$ in J . Since γ is not vertical near $a = 0$, we find a sequence $(t_\delta) \subset J$ such that $t_\delta \searrow 0$, $y(t_\delta) \searrow 0$, $x'(t_\delta) \searrow 0$ as $\delta \rightarrow 0$ and either $x'(t_\delta) > 0$ or $x'(t_\delta) < 0$ for any δ . Again it suffices to consider the case $x'(t_\delta) > 0$, as the other is dealt with by a mirrored construction.

We aim to connect γ at $t = t_\delta$ to a circle with unit speed parametrisation (k_δ, l_δ) given by

$$k_\delta(t) = r_\delta - r_\delta \cos\left(\frac{t - a_\delta}{r_\delta}\right) + m_\delta \quad \text{and} \quad l_\delta(t) = r_\delta \sin\left(\frac{t - a_\delta}{r_\delta}\right) + n_\delta$$

with radius r_δ , shifts m_δ, n_δ , and parameter shift a_δ to be found; see Figure 4.1(1). At $t = a_\delta$ we have $(k'_\delta, l'_\delta) = (0, 1)$, thus the circle can be connected to a vertical line segment provided that $n_\delta = l_\delta(a_\delta) > 0$. At $t = t_\delta$ we have to satisfy the conditions

$$\begin{aligned} k'_\delta(t_\delta) &= x'(t_\delta), & k_\delta(t_\delta) &= x(t_\delta), \\ l'_\delta(t_\delta) &= y'(t_\delta), & l_\delta(t_\delta) &= y(t_\delta) \end{aligned}$$

in order to match end points and derivatives of γ and the circle. A short computation shows that

$$\frac{t_\delta - a_\delta}{r_\delta} = \arctan \frac{x'(t_\delta)}{y'(t_\delta)} \quad (4.11)$$

and

$$x(t_\delta) = r_\delta(1 - y'(t_\delta)) + m_\delta, \quad y(t_\delta) = r_\delta x'(t_\delta) + n_\delta.$$

These equations determine a_δ , m_δ , and r_δ in terms of the given t_δ , $\gamma(t_\delta)$, $\gamma'(t_\delta)$, and the still free n_δ ; choosing $n_\delta = y(t_\delta)/2 > 0$, we obtain

$$r_\delta = \frac{n_\delta}{x'(t_\delta)}, \quad m_\delta = x(t_\delta) - n_\delta \frac{1 - y'(t_\delta)}{x'(t_\delta)}, \quad \text{and} \quad a_\delta = t_\delta - n_\delta \frac{\arctan(x'/y')(t_\delta)}{x'(t_\delta)}.$$

The shift n_δ tends to 0 as $\delta \rightarrow 0$ by definition, and $m_\delta \rightarrow 0$ is a consequence of

$$\frac{1 - y'(t_\delta)}{x'(t_\delta)} = \frac{1 - y'(t_\delta)}{\sqrt{1 - y'(t_\delta)^2}} = \sqrt{\frac{1 - y'(t_\delta)}{1 + y'(t_\delta)}} \rightarrow 0.$$

Moreover, $0 \leq \arctan z \leq z$ for $z \geq 0$ and $1/2 \leq y'(t_\delta) \leq 1$ imply

$$t_\delta \geq a_\delta \geq t_\delta - \frac{y(t_\delta)}{2y'(t_\delta)} \geq t_\delta - y(t_\delta) \geq 0,$$

hence $a_\delta \rightarrow 0$ as $\delta \rightarrow 0$ follows.

With these circles we define a local approximation for γ by

$$\gamma_\delta(t) = \begin{cases} \gamma(t) & \text{if } t_\delta \leq t, \\ (k_\delta(t), l_\delta(t)) & \text{if } a_\delta \leq t < t_\delta, \\ (m_\delta, t + n_\delta - a_\delta) & \text{if } a_\delta - n_\delta \leq t < a_\delta. \end{cases}$$

Here the third part is a vertical line segment of unit speed that connects (m_δ, n_δ) at $t = a_\delta$ with $(m_\delta, 0)$ at $t = a_\delta - n_\delta$. Clearly, γ_δ belongs to $W^{2,2}((a_\delta - n_\delta, t_0); \mathbb{R}^2)$. Since the vertical line and the circle segment vanish in the limit $\delta \rightarrow 0$, we have pointwise convergence of γ_δ and γ'_δ to γ and γ' , respectively. Moreover, the area of $M_\delta(a_\delta - n_\delta, t_0)$ converges to the area of $M(0, t_0)$.

On the vertical segment both principal curvatures and all curvature integrals are zero. On the circle segment we have $|\kappa_{1,\delta}| = 1/r_\delta$ and thus

$$\begin{aligned} \frac{1}{2\pi} \int_{M_\delta(a_\delta, t_\delta)} \kappa_{1,\delta}^2 d\mu_\delta &= \int_{a_\delta}^{t_\delta} \frac{1}{r_\delta^2} (r_\delta \sin((t - a_\delta)/r_\delta) + n_\delta) dt \\ &\leq 2 \frac{t_\delta - a_\delta}{r_\delta} \end{aligned} \quad (4.12)$$

using $n_\delta/r_\delta = x'(t_\delta) \in (0, 1]$; for the second principal curvature $\kappa_{2,\delta} = x'_\delta/y_\delta$ we compute

$$\begin{aligned} \frac{1}{2\pi} \int_{M_\delta(a_\delta, t_\delta)} \kappa_{2,\delta}^2 d\mu_\delta &= \int_{a_\delta}^{t_\delta} \frac{1}{r_\delta} \frac{\sin((t - a_\delta)/r_\delta)^2}{\sin((t - a_\delta)/r_\delta) + n_\delta/r_\delta} dt \\ &\leq \frac{t_\delta - a_\delta}{r_\delta}. \end{aligned} \quad (4.13)$$

Due to (4.11) and $x'(t_\delta) \rightarrow 0$, both (4.12) and (4.13) tend to 0 as $\delta \rightarrow 0$, hence we obtain

$$\int_{M_\delta(a_\delta - n_\delta, t_0)} |B_\delta|^2 d\mu_\delta \rightarrow \int_{M(0, t_0)} |B|^2 d\mu.$$

As u is constant in $J = (0, t_0)$, we may define $u_\delta(t) = u|_J$ for $t \in [a_\delta - n_\delta, t_0]$ and $u_\delta(t) = u(t)$ for $t \in \omega, t > t_0$. As with the curves, u_δ converges to u pointwise, and we obtain

$$\begin{aligned} \int_{M_\delta(a_\delta - n_\delta, t_0)} k(u_\delta) (H_\delta - H_s(u_\delta))^2 + k_G(u_\delta) K_\delta d\mu_\delta \\ \rightarrow \int_{M(0, t_0)} k(u) (H - H_s(u))^2 + k_G(u) K d\mu. \end{aligned}$$

Note in particular, that both Gauss curvature integrals are equal, because they depend only on the tangent angle of γ_δ or γ at t_δ and $a_\delta - n_\delta$ or 0, respectively.

In order to fit the above construction into the neighbouring components of ω , γ_δ and the rest of the original curve γ have to be shifted in x and t . These shifts, however, vanish as $\delta \rightarrow 0$ and thus do not disturb the proved convergences. Applying the above procedure to the boundaries of each component and gluing together the resulting segments gives a membrane $(\gamma_\delta, u_\delta)$ defined on some interval I_δ , which converges to I as $\delta \rightarrow 0$ in the sense that the boundary points converge.

It remains to correct the area and the phase field constraint as well as the parameter interval. For the area constraint we fix $J \Subset \{y > 0\} \setminus S_u$ such that apart from the shifts γ is unchanged in J for all small δ , $x' > 0$ or $x' < 0$ in \bar{J} , and such that $M_\gamma(J)$ is not part of a catenoid. Such an interval exists, because otherwise γ restricted to any component of $\{y > 0\}$ would consist only of vertical lines and catenary segments, which is impossible for a C^1 -curve that starts and ends on the x -axis. After an additional parameter shift of γ_δ we may assume that $\gamma_\delta(J) = \gamma(J)$ up to an x -shift. Let $f \in C_c^\infty(J; \mathbb{R}^2)$ and consider the curve $\tilde{\gamma}_{\delta, \alpha} = \gamma_\delta + \alpha f$, whose corresponding surface of revolution has the area

$$\mathcal{A}_{\tilde{\gamma}_{\delta, \alpha}} = \mathcal{A}_{\gamma_\delta} + \mathcal{A}_{\tilde{\gamma}_{\delta, \alpha}}(J) - \mathcal{A}_\gamma(J).$$

Then the requirement $\mathcal{A}_\gamma = \mathcal{A}_{\tilde{\gamma}_{\delta, \alpha}}$ is equivalent to

$$\mathcal{A}_{\tilde{\gamma}_{\delta, \alpha}}(J) - \mathcal{A}_\gamma(J) = \mathcal{A}_\gamma - \mathcal{A}_{\gamma_\delta}. \quad (4.14)$$

The left hand side of (4.14) equals 0 for $\alpha = 0$ and depends continuously on α ; it is strictly positive for one sign of α and strictly negative for the other, since $\gamma(J)$ is not a catenoid segment and $M_\gamma(J)$ not stationary for the area. The right hand side of (4.14) vanishes as $\delta \rightarrow 0$, hence for all sufficiently small δ there is an α_δ such that (4.14) holds and $\alpha_\delta \rightarrow 0$ as $\delta \rightarrow 0$. Thus, gluing together $\tilde{\gamma}_{\delta, \alpha_\delta}$ instead of γ_δ accounts for the area constraint at the cost of violating the constant speed requirement. The latter, however, is fixed by a global reparametrisation, which also gives a membrane defined on I . Since $I_\delta \rightarrow I$ as $\delta \rightarrow 0$ and the perturbations from the area recovery vanishes with $\alpha_\delta \rightarrow 0$ in any function space, these reparametrisations converge to the identity in $W^{2,2}$ and the convergences of curvature integrals, curves, and phase fields still hold. Using the uniform bounds on γ_δ , γ'_δ , and u_δ we obtain convergence of γ_δ in $W^{1,p}(I; \mathbb{R}^2)$ and u_δ in $L^p(I)$. Since number and height of interfaces are not affected, the interface energy remains unchanged.

The phase integral constraint is easily recovered by moving an existing interface slightly or introducing one or finitely many new ones at a height that vanishes with $\delta \rightarrow 0$. \square

Remark. The construction in the proof of Lemma 4.5 can also be done at ∂I . Hence, the Lemma comprises the result that any $\gamma \in \mathcal{C}$ can be approximated by curves from $\mathcal{C} \cap W^{2,2}(I; \mathbb{R}^2)$.

The next step is to find a recovery sequence for membranes (γ, u) as constructed in Lemma 4.5. To this end, let $s \in \{y = 0\}$ and fix $J \Subset I$ such that $\bar{J} \cap (\{y = 0\} \cup S_u) = \{s\}$ and γ is a vertical line in $J \cap \{t > s\}$ and $J \cap \{t < s\}$. For simplicity of notation we assume again $s = 0$, $\gamma(0) = (0, 0)$ and $|\gamma'| = 1$.

A δ -catenoid is the surface generated by a δ -catenary whose unit speed parametrisation $c_\delta = (i_\delta, j_\delta) \in C^\infty(\mathbb{R}; \mathbb{R}^2)$ is given by

$$i_\delta(t) = \delta \operatorname{arcsinh} \frac{t}{\delta}, \quad j_\delta(t) = \sqrt{\delta^2 + t^2}.$$

Its principal curvatures are

$$-\kappa_{1,\delta}(t) = \kappa_{2,\delta}(t) = \frac{\delta}{\delta^2 + t^2},$$

and thus we have

$$\int_{M_{c_\delta}(a,b)} |B_\delta|^2 d\mu_\delta = 4\pi \int_a^b \frac{\delta^2}{(\delta^2 + t^2)^2} \sqrt{\delta^2 + t^2} dt = 4\pi \left. \frac{t}{\sqrt{\delta^2 + t^2}} \right|_a^b \leq 8\pi \quad (4.15)$$

for any $\delta > 0$ and all $a, b \in \mathbb{R}$, $a < b$. Since the δ -catenary satisfies $c_\delta(0) = (0, \delta)$ and $c'_\delta(0) = (1, 0)$, it suffices to study a construction for $J \cap \{t \geq 0\}$ and join it with its mirrored counterpart in $J \cap \{t \leq 0\}$.

Lemma 4.6. *Assume that γ is a vertical line segment in $J = (0, t_0)$, that is $\gamma(t) = (0, t)$ in J . Then for all sufficiently small δ depending only on γ there is a curve $\gamma_\delta = (x_\delta, y_\delta) \in W^{2,2}(J; \mathbb{R}^2)$ such that*

- γ_δ satisfies $y_\delta > 0$, $x'_\delta \geq 0$ in J and there is $J_\delta = (0, t_\delta)$ for some $t_\delta \rightarrow 0$ as $\delta \rightarrow 0$ such that γ_δ is a vertical line segment in $J \setminus J_\delta$ and

$$|\gamma'_\delta| = \begin{cases} 1 & \text{if } t \in J \setminus \tilde{J}, \\ 1 + r_\delta & \text{if } t \in \tilde{J}, \end{cases}$$

where $r_\delta \rightarrow 0$ in $W^{1,2}(\tilde{J}) \cap C_c^0(\tilde{J})$ as $\delta \rightarrow 0$ and $\tilde{J} \subset J \setminus J_\delta$;

- at the end points of $\gamma_\delta(J)$ we have

$$\gamma_\delta(0) = (0, \delta), \quad \gamma'_\delta(0) = (1, 0), \quad \gamma_\delta(t_0) = (x(t_0) + o(1), y(t_0)), \quad \gamma'_\delta(t_0) = \gamma'(t_0);$$

- $\gamma_\delta \rightarrow \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for any $p \in [1, \infty)$ as $\delta \rightarrow 0$;
- $\mathcal{A}_\delta(J) = \mathcal{A}(J) + o(1)$ and $\int_{M_\delta(J)} u d\mu_\delta = \int_{M_\gamma(J)} u d\mu + o(1)$ as $\delta \rightarrow 0$;
- $\sup_{\delta > 0} \int_{M_\delta(J)} |B_\delta|^2 d\mu_\delta < \infty$; and
- $\int_{M_\delta(J)} k(u)(H_\delta - H_s(u))^2 d\mu_\delta \rightarrow \int_{M_\gamma(J)} k(u)(H - H_s(u))^2 d\mu$ as $\delta \rightarrow 0$.

Proof. With s_δ and t_δ to be determined, we replace γ by a δ -catenary in some interval $[0, s_\delta)$ and a segment of a circle of radius 1 in $[s_\delta, t_\delta)$, which connects the catenary and the shifted original curve; see Figure 4.1(3). Writing

$$k_\delta(t) = \sin(t - b_\delta) + \hat{k}_\delta \quad \text{and} \quad l_\delta(t) = -\cos(t - b_\delta) + \hat{l}_\delta$$

for the coordinates of the circle and fixing δ and s_δ for the moment, we aim to determine t_δ , b_δ , \hat{k}_δ , \hat{l}_δ , and shifts m_δ , n_δ such that

$$\gamma_\delta(t) = \begin{cases} c_\delta(t) & \text{if } 0 \leq t < s_\delta, \\ (k_\delta(t), l_\delta(t)) & \text{if } s_\delta \leq t < t_\delta, \\ (m_\delta, t + n_\delta) & \text{if } t_\delta \leq t \end{cases}$$

is continuously differentiable at s_δ and t_δ . The corresponding conditions are

$$\begin{aligned} k'_\delta(s_\delta) &= i'_\delta(s_\delta), & l'_\delta(s_\delta) &= j'_\delta(s_\delta), & k_\delta(s_\delta) &= i_\delta(s_\delta), & l_\delta(s_\delta) &= j_\delta(s_\delta), \\ k'_\delta(t_\delta) &= 0, & l'_\delta(t_\delta) &= 1, & k_\delta(t_\delta) &= m_\delta, & l_\delta(t_\delta) &= t + n_\delta, \end{aligned}$$

and a short calculation shows

$$\begin{aligned} b_\delta &= s_\delta - \arctan(s_\delta/\delta), & t_\delta &= \pi/2 + b_\delta, \\ \hat{k}_\delta &= i_\delta(s_\delta) - \sin(s_\delta - b_\delta), & \hat{l}_\delta &= j_\delta(s_\delta) + \cos(s_\delta - b_\delta), \\ m_\delta &= 1 + \hat{k}_\delta, & n_\delta &= \hat{l}_\delta - t_\delta. \end{aligned}$$

If we let $s_\delta = \delta^\beta$ for some $\beta \in (0, 3/4)$, we find

$$t_\delta \sim \delta^\beta + \delta^{1-\beta}, \quad m_\delta \sim \delta \ln \delta^{\beta-1}, \quad n_\delta \sim \delta^\beta + \delta^{1-\beta},$$

that is, the catenary and circle vanish in the limit $\delta \rightarrow 0$, and therefore $\gamma_\delta \rightarrow \gamma$ in $W^{1,p}(J; \mathbb{R}^2)$ for $p \in [1, \infty)$ and $\mathcal{A}_{\gamma_\delta}(J) \rightarrow \mathcal{A}_\gamma(J)$. A more precise estimate shows $\mathcal{A}_{\gamma_\delta}(J) = \mathcal{A}_\gamma(J) + O(\delta + \delta^{2\beta} + \delta^{2-2\beta})$ and the same order for the error in the phase integral constraint.

The principal curvatures of the circle segment are

$$\kappa_1 = -1 \quad \text{and} \quad \kappa_2 = \frac{\cos(t - b_\delta)}{\hat{l}_\delta - \cos(t - b_\delta)},$$

thus the second fundamental form is estimated by

$$\begin{aligned} \frac{1}{2\pi} \int_{M_{\gamma_\delta}(s_\delta, t_\delta)} |B_\delta|^2 d\mu_\delta &\leq \left(1 + \frac{\cos^2(s_\delta - b_\delta)}{j_\delta(s_\delta)}\right) \cdot (t_\delta - s_\delta) \\ &= \left(1 + \frac{\delta^2}{(\delta^2 + \delta^{2\beta})^{3/2}}\right) \cdot \left(\pi/2 - \arctan \delta^{\beta-1}\right) \\ &\sim \left(1 + \delta^{2-3\beta}\right) \cdot \delta^{1-\beta} = \delta^{1-\beta} + \delta^{3-4\beta}, \end{aligned}$$

which tends to 0 as $\delta \rightarrow 0$ due to $0 < \beta < 3/4$. By (4.15) the second fundamental forms of M_{γ_δ} are thus uniformly bounded in L^2 . Similarly, the integral of $H_{\gamma_\delta}^2$ over the circle segments vanishes in the limit $\delta \rightarrow 0$, and since $H_{\gamma_\delta}(t) = 0$ for $t < s_\delta$ and $H_{\gamma_\delta}(t) = H_\gamma(t)$ for $t > t_\delta$, we obtain

$$\int_{M_{\gamma_\delta}(J)} k(u)(H_\delta - H_s(u))^2 d\mu_{\gamma_\delta} \rightarrow \int_{M_\gamma(J)} k(u)(H - H_s(u))^2 d\mu$$

as $\delta \rightarrow 0$.

To dispose of the shift n_δ in y -direction, fix $\tilde{J} \Subset J$ with $t_\delta < \inf \tilde{J}$ for all sufficiently small δ and a function $f \in C_c^\infty(\tilde{J})$ with $\int_J f dt = 1$. The perturbed curve $\tilde{\gamma}_\delta = (\tilde{x}_\delta, \tilde{y}_\delta) = (x_\delta, y_\delta - n_\delta F)$, where $F(t) = \int_0^t f(s) ds$, has the desired end point y -coordinate

$$\tilde{y}_\delta(t_0) = y_\delta(t_0) - n_\delta = y(t_0).$$

Since $|n_\delta| \rightarrow 0$ as $\delta \rightarrow 0$, the perturbation vanishes in any function space to which γ_δ belongs. The second fundamental form is still uniformly bounded in $L^2(J)$, because for all small δ the perturbation is supported in a vertical line segment of γ_δ , where both principal curvatures are equal to 0. Moreover, we have $\inf_{\tilde{J}} \tilde{y}_\delta > 0$ and the error in the area constraint in J and in the phase integral constraint are of the same order as above. \square

Now it is easy to see why $k_G(u)$ has to be adapted near points on the axis of revolution, even if u does not have a jump there. Consider for instance a surface M_γ consisting of two balls connected at the axis of revolution and assume that $k_G(u) \equiv -1$. Then $\int_{M_\gamma} k_G K d\mu = -8\pi$, while for any approximation M_δ with one component we have $\int_{M_\delta} k_G K_\delta d\mu_\delta = -4\pi$. Since the mean curvature integral converges, the total curvature energy drops in the limit $\varepsilon \rightarrow 0$. Changing the phase field such that $k_G(u_\varepsilon) = 0$ compensates for this effect.

In the following corollary we combine all previous constructions and apply the additional phase field change. This finishes the proof of the upper bound.

Corollary 4.7. *Let $(\gamma, u) \in \mathcal{C} \times \mathcal{P}$ have finitely many interfaces and vertical line segments near component boundaries in I . Then there are $(\gamma_\varepsilon, u_\varepsilon) \in \mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ such that $\gamma_\varepsilon \rightarrow \gamma$ in $W^{1,1}(I; \mathbb{R}^2)$, $u_\varepsilon \rightarrow u$ in $L^1(I)$ and $\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \rightarrow \mathcal{E}(\gamma, u)$ as $\varepsilon \rightarrow 0$.*

Proof. Let $\{y = 0\} \cap I = \{s_1, \dots, s_n\}$ where $n = N_\gamma - 1$. We employ Lemma 4.6 and its mirrored version successively for each s_k , taking the global shifts in x -direction into account. Since the parameter $\delta > 0$ in Lemma 4.6 is independent of ε , we can choose it so small that the curve replacement, apart from x -shifts and the small perturbation in Lemma 4.6, takes place in intervals $J_{k,\varepsilon}$ around s_k of length at most ε and with area and phase constraint error bounded by $\sqrt{\varepsilon}$. The result for sufficiently small ε is a sequence (γ_ε) that converges to γ in $W^{1,p}(I; \mathbb{R}^2)$ for any $p \in [1, \infty)$ and satisfies

$$\int_{M_\varepsilon} k(u)(H_\varepsilon - H_s(u))^2 d\mu_\varepsilon \rightarrow \int_{M_\gamma} k(u)(H - H_s(u))^2 d\mu \quad \text{as } \varepsilon \rightarrow 0. \quad (4.16)$$

We construct u_ε by first replacing u around S_u with the local recovery sequences from Section 4.3.2. Additionally, we set $u_\varepsilon = 0$ in each $J_{k,\varepsilon}$ and at $\partial J_{k,\varepsilon}$ we make a transition to ± 1 exactly as in Section 4.3.2. The phase field energy of $J_{k,\varepsilon}$ is

$$2\pi \int_{J_{k,\varepsilon}} \frac{1}{\varepsilon} W(0) y_\varepsilon dt \leq 2\pi W(0) \sup_{J_{k,\varepsilon}} y_\varepsilon,$$

and the costs of the transitions near $\partial J_{k,\varepsilon}$ are bounded by $2\pi\sigma \sup_{\partial J_{k,\varepsilon}} y_\varepsilon$ as shown in Lemma 4.4. By construction of γ_ε both vanish as $\varepsilon \rightarrow 0$. Moreover

$$\int_{M_\varepsilon} k_G(u_\varepsilon) K_\varepsilon d\mu_\varepsilon = \int_{M_\varepsilon(I \setminus \cup J_{k,\varepsilon})} k_G(u_\varepsilon) K_\varepsilon d\mu_\varepsilon \rightarrow \int_{M_\gamma} k_G(u) K d\mu,$$

since $u_\varepsilon \rightarrow u$ in $L^1(I)$, $K_\varepsilon = K$ outside $\cup J_{k,\varepsilon}$, and $\int_M |K| d\mu$ is finite. Also, (4.16) still holds with u replaced by u_ε on the left hand side. Hence, we find $\mathcal{E}_\varepsilon(\gamma_\varepsilon, u_\varepsilon) \rightarrow \mathcal{E}(\gamma, u)$ as $\varepsilon \rightarrow 0$.

Finally, to obtain a membrane in $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ we once more have to correct the constraints. For the area this is done as in Lemma 4.5, while the error in the phase integral, which is introduced by the constructions at the axis of revolution, can be corrected as in Lemma 4.4, as it is of order $\sqrt{\varepsilon}$. The result is a membrane $(\gamma_\varepsilon, u_\varepsilon)$ that satisfies all conditions of $\mathcal{C}_\varepsilon \times \mathcal{P}_\varepsilon$ except for the constant speed requirement. However, since by construction $|\gamma'_\varepsilon| = 1 + o(1)$ and the perturbation vanishes in $W^{1,2}$, the constant speed reparametrisations converge to the identity in $W^{2,2}(I)$ and the properties of $(\gamma_\varepsilon, u_\varepsilon)$ carry over to reparametrised curve and phase field. \square

5 Some generalisations

We conclude the paper with some extensions of Theorem 3.1. First of all, the proof is easily adapted to non-symmetric potentials W . In this case, one considers the complete optimal profile p in Section 4.3.2 and uses the appropriate side in the connections to regions $\{u_\varepsilon = 0\}$ in Corollary 4.7. One may also consider potentials like $W(u) = (1 - u)^2$ and drop the phase integral constraint for u_ε . Then there is only one lipid phase, and u_ε is merely an auxiliary variable that allows the recovery of topological changes at the axis of revolution in the limit.

The constraint of prescribed area for the approximate setting can be relaxed to

$$0 < \inf_{\gamma \in \mathcal{C}_\varepsilon} \mathcal{A}_\gamma \leq \sup_{\gamma \in \mathcal{C}_\varepsilon} \mathcal{A}_\gamma < \infty,$$

and the arguments for equi-coercivity and lower bound still apply. It can be incorporated as penalty term in the energy, for instance by $(\mathcal{A}_\gamma - A_0)^2/\varepsilon$ or any other scale of ε , because we have recovered it exactly. In the same way, the phase integral constraint can be replaced by a penalty term. Other constraints that change continuously under the convergence proved in Lemmas 4.1 and 4.2 can also be imposed, for instance on the enclosed volume $\mathcal{V}_\gamma(M) = \pi \int_{M_\gamma} x'y^2 dt$. Of course, constraints have to be compatible, so that the set of admissible membranes is non-empty.

The arguments in Section 4 also apply to open surfaces of revolution generated by curves $\gamma = (x, y): I \rightarrow \mathbb{R} \times \mathbb{R}_{>0}$ with prescribed boundary conditions for γ and γ' at ∂I . The curve length is then controlled by energy, area, and boundary conditions due to

$$2\pi\mathcal{L}_\gamma \leq \int_{M_\gamma} |H| d\mu + 2\pi\tilde{\varphi}y|_{\partial I},$$

where $\tilde{\varphi}$ is the tangent angle as in the proof of Lemma 2.3. Note that the boundary condition for γ' at ∂I is preserved as $\varepsilon \rightarrow 0$ because $y > 0$ at ∂I . Furthermore, since $\|\tilde{\varphi}\|_\infty \leq \pi/2$, it is also possible to weaken the boundary conditions to requiring a uniform L^∞ -bound on y at ∂I . Such a bound can for instance be derived from uniformly bounded energy $\mathcal{E}_\varepsilon + \mathcal{G}$, where

$$\mathcal{G}(\gamma) = \bar{\sigma} \int_{M_\gamma(\partial I)} d\mathcal{H}^1 = 2\pi\bar{\sigma} \sum_{s \in \partial I} y(s)$$

with a constant line tension $\bar{\sigma}$. Since \mathcal{G} is continuous with respect to curve convergence in C^0 , its presence does not influence the Γ -convergence. The limit energy $\mathcal{E} + \mathcal{G}$ models open lipid membranes; see for instance [20, 27, 28] for experimental observations, modelling and numerical simulations of single-phase open membranes, respectively.

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